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A q-analogue of $\widehat{\mathfrak{gl}}_3$ hierarchy and q-Painlevé VI

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Abstract
A q-analogue of the $\widehat{\mathfrak{gl}}_3$ Drinfel'd–Sokolov hierarchy is proposed as a reduction of the q-KP hierarchy. Applying a similarity reduction and a q-Laplace transformation to the hierarchy, one can obtain the q-Painlevé VI equation proposed by Jimbo and Sakai.

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Dedicated to Professors Junkichi Satsuma and Basil Grammaticos on the occasion of the 60th birthday.

1. Introduction

In the preceding work [1], we established a relationship between the generic Painlevé VI equation and the $\widehat{\mathfrak{gl}}_3$ Drinfel'd–Sokolov hierarchy that contains the three-wave resonant system. Our approach is based on a similarity reduction of the generalized Drinfel'd–Sokolov hierarchy that has been discussed in [2]. We remark that Conte, Grundland and Musette also discussed a reduction from the three-wave resonant system to the generic Painlevé VI [3].

On the other hand, q-difference soliton equations have been discussed by several researchers [4–10]. In [9], Kajiwara, Noumi and Yamada discussed a q-analogue of a similarity reduction from the q-KP hierarchy to q-Painlevé equations. The main purpose of the present paper is to obtain a q-analogue of the Painlevé VI equation as a similarity reduction of the multi-component q-KP hierarchy.

In [11], Jimbo and Sakai proposed a q-difference analogue of the sixth Painlevé equation (q-P_{VI}), which is a coupled system of q-difference equations:

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$$\begin{cases} \dfrac{y(t)y(qt)}{a_3a_4} = \dfrac{\{z(qt) - tb_1\}\{z(qt) - tb_2\}}{\{z(qt) - b_3\}\{z(qt) - b_4\}}, \\ \dfrac{z(t)z(qt)}{b_3b_4} = \dfrac{\{y(t) - ta_1\}\{y(t) - ta_2\}}{\{y(t) - a_3\}\{y(t) - a_4\}}, \end{cases} \tag{1}$$

where the parameters $a_j, b_j (j = 1, 2, 3, 4)$ obey the constraint

$$\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}. \tag{2}$$

These equations are obtained from a connection preserving deformation of a linear q-difference equation,

$$Y(q\zeta, t) = \mathcal{A}(\zeta; t)Y(\zeta, t), \qquad \mathcal{A}(\zeta; t) \stackrel{\text{def}}{=} \mathcal{A}_0(t) + \mathcal{A}_1(t)\zeta + \mathcal{A}_2(t)\zeta^2, \tag{3}$$

$$Y(\zeta, qt) = \frac{\zeta(\zeta I + \mathcal{B}_0(t))}{(\zeta - qta_1)(\zeta - qta_2)}Y(\zeta, t), \tag{4}$$

where $Y(\zeta, t)$ is a (2×2)-matrix-valued function with respect to ζ and t. The coefficient matrices $\mathcal{A}_j(t)(j = 0, 1, 2)$ are assumed to satisfy the conditions

$$\begin{gathered} \mathcal{A}_2(t) = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \quad \text{eigenvalues of } \mathcal{A}_0(t) \text{ are } t\theta_1, t\theta_2, \\ \det \mathcal{A}(\zeta, t) = \kappa_1\kappa_2(\zeta - ta_1)(\zeta - ta_2)(\zeta - a_3)(\zeta - a_4), \end{gathered} \tag{5}$$

where the parameters $\kappa_i, \theta_i (i = 1, 2)$ are given by

$$\kappa_1 = \frac{1}{qb_3}, \qquad \kappa_2 = \frac{1}{b_4}, \qquad \theta_1 = \frac{a_1a_2}{b_1}, \qquad \theta_2 = \frac{a_1a_2}{b_2}. \tag{6}$$

The variables $y(t), z(t)$ of the q-$\mathrm{P_{VI}}$ are related to the coefficient matrix $\mathcal{A}(\zeta; t)$ as follows:

$$(\mathcal{A}(\zeta = y(t); t))_{12} = 0, \tag{7}$$

$$z(t) = \frac{(y - ta_1)(y - ta_2)}{q(\mathcal{A}(\zeta = y; t))_{11}} = \frac{(\mathcal{A}(\zeta = y; t))_{22}}{q\kappa_1\kappa_2(y - a_3)(y - a_4)}, \tag{8}$$

where $(M)_{ij}$ denotes the (i, j) component of a matrix M.

In the following section, we introduce a q-analogue of the $\widehat{\mathfrak{gl}}_N$ hierarchy as a reduced case of the multi-component q-KP hierarchy based on the work [10]. We will show that the q-Painlevé VI can be obtained as a similarity reduction of the q–$\widehat{\mathfrak{gl}}_3$ hierarchy.

2. A q-analogue of $\widehat{\mathfrak{gl}}_3$ hierarchy

Throughout the paper, we assume $|q| > 1$ unless mentioned otherwise. We will use the following notations:

(shift operator) $\mathrm{e}^{m\partial_s} f(s) = f(s + m)$,
(q-shift operator) $(T_q(z))^m f(z) = f(q^m z)$,
(q-difference operator) $\mathcal{D}_q(z) f(z) = \dfrac{1 - T_q(z)}{z} f(z) = \dfrac{f(z) - f(qz)}{z}$,
(q-shifted factorial) $(z; q^{-1})_n = \displaystyle\prod_{j=0}^{n-1}(1 - q^{-j}z)$, $(z; q^{-1})_\infty = \displaystyle\prod_{j=0}^{\infty}(1 - q^{-j}z)$.

To describe a q-analogue of the multi-component KP hierarchy [12], we define the Sato–Wilson operators,

$$W(e^{\partial_s}; s, \underline{x}) = I + W_1 e^{-\partial_s} + W_2 e^{-2\partial_s} + \cdots,$$
$$\bar{W}(e^{\partial_s}; s, \underline{x}) = \bar{W}_0 + \bar{W}_1 e^{\partial_s} + \bar{W}_2 e^{2\partial_s} + \cdots. \tag{9}$$

The coefficients $W_i = W_i(s; \underline{x})(i = 1, 2, \ldots)$, $\bar{W}_j = \bar{W}_j(s; \underline{x})(j = 0, 1, 2, \ldots)$ are $(N \times N)$-matrix-valued functions that depend on a discrete variable s and a set of parameters $\underline{x} = \{x_1^{(k)}, x_2^{(k)}, \ldots (k = 1, \ldots, N)\}$. We assume that $\bar{W}_0$ is invertible.

For a difference operator $A(e^{\partial_s}) = \sum_n A_n e^{n\partial_s}$, we denote by $[A(e^{\partial_s})]_{\geqslant 0}$ the projection to the non-negative part: $[A(e^{\partial_s})]_{\geqslant 0} = \sum_{n\geqslant 0} A_n e^{n\partial_s}$. We define a q-analogue of the Sato equation as

$$\mathcal{D}_q\big(x_n^{(k)}\big)\widetilde{W} = \big[\big(T_q\big(x_n^{(k)}\big)W\big)I_k e^{n\partial_s} W^{-1}\big]_{\geqslant 0}\widetilde{W} - \big(T_q\big(x_n^{(k)}\big)\widetilde{W}\big)I_k e^{n\partial_s}, \tag{10}$$

where $\widetilde{W} = W$ or $\bar{W}$, and $I_k = [\delta_{ij}\delta_{ik}]_{1\leqslant i,j\leqslant N}$. We remark the hierarchy defined above is slightly different from that of [10].

Proposition 1 (Scaling symmetry)**.** *For a constant $\lambda \in \mathbb{C}^\times$, we define W_λ and $\bar{W}_\lambda$ as*

$$W_\lambda(e^{\partial_s}; s, \underline{x}) \stackrel{\text{def}}{=} \lambda^{s+D(\alpha)} \circ W(e^{\partial_s}; s, \underline{x}_\lambda) \circ \lambda^{-s-D(\alpha)}, \tag{11}$$

$$\bar{W}_\lambda(e^{\partial_s}; s, \underline{x}) \stackrel{\text{def}}{=} \lambda^{s+D(\alpha)} \circ \bar{W}(e^{\partial_s}; s, \underline{x}_\lambda) \circ \lambda^{-s-D(\beta)}, \tag{12}$$

where $D(\alpha) = \mathrm{diag}[\alpha_1, \ldots, \alpha_N]$, $D(\beta) = \mathrm{diag}[\beta_1, \ldots, \beta_N]$ are constant matrices and $\underline{x}_\lambda = \{\lambda x_1^{(k)}, \lambda^2 x_2^{(k)}, \ldots (k = 1, \ldots, N)\}$. If W and $\bar{W}$ solve the q-Sato equation (10), so do W_λ and $\bar{W}_\lambda$.

Proposition 1 can be checked by a direct calculation.

We define formal Baker–Akhiezer functions,

$$\Psi_q^{(\infty)}(z; s, \underline{x}) = W(z; s, \underline{x})\Psi_{q,0}^{(\infty)}(z; s, \underline{x}), \tag{13}$$

$$\Psi_{q,0}^{(\infty)}(z; s, \underline{x}) = z^{s+D(\alpha)} \prod_{j\geqslant 1} \mathrm{diag}\Big[\big(z^j x_j^{(1)} q^{-1}; q^{-1}\big)_\infty, \ldots, \big(z^j x_j^{(N)} q^{-1}; q^{-1}\big)_\infty\Big], \tag{14}$$

$$\Psi_q^{(0)}(z; s, \underline{x}) = \bar{W}(z; s, \underline{x})\Psi_{q,0}^{(0)}(z; s, \underline{x}), \tag{15}$$

$$\Psi_{q,0}^{(0)}(z; s, \underline{x}) = z^{s+D(\beta)} \prod_{j\geqslant 1} \mathrm{diag}\Big[\big(z^j x_j^{(1)} q^{-1}; q^{-1}\big)_\infty, \ldots, \big(z^j x_j^{(N)} q^{-1}; q^{-1}\big)_\infty\Big], \tag{16}$$

where we have assumed that $|q| > 1$ for convergence. From (10), it follows that both $\Psi_q^{(\infty)}(z; s, x)$ and $\Psi_q^{(0)}(z; s, x)$ satisfy the same q-difference equation of the form

$$\mathcal{D}_q\big(x_n^{(k)}\big)\Psi_q(z; s, \underline{x}) = \big[\big(T_q\big(x_n^{(k)}\big)W\big)I_k e^{n\partial_s} W^{-1}\big]_{\geqslant 0}\Psi_q(z; s, \underline{x}). \tag{17}$$

We now impose the condition,

$$W(e^{\partial_s}; s+1, \underline{x}) = W(e^{\partial_s}; s, \underline{x}), \qquad \bar{W}(e^{\partial_s}; s+1, \underline{x}) = \bar{W}(e^{\partial_s}; s, \underline{x}). \tag{18}$$

If a difference operator $A(e^{\partial_s}; s)$ satisfies the condition $A(e^{\partial_s}; s+1) = A(e^{\partial_s}; s)$, the correspondence

$$A(e^{\partial_s}; s) = \sum_{n\in\mathbb{Z}} A_n(s) e^{n\partial_s} \quad \leftrightarrow \quad A(z; s) = \sum_{n\in\mathbb{Z}} A_n(s) z^n \tag{19}$$

preserves sums, products and commutators [12]. Here z is used as a formal indeterminate. The q-Sato equation (10) then takes the following form:

$$\mathcal{D}_q\left(x_n^{(k)}\right)\widetilde{W} = C_n^{(k)}\widetilde{W} - z^n\left(T_q\left(x_n^{(k)}\right)\widetilde{W}\right)I_k, \qquad \widetilde{W} = W, \bar{W}, \tag{20}$$

$$C_n^{(k)}(z;\underline{x}) = \left[z^n\left(T_q\left(x_n^{(k)}\right)W\right)I_k W^{-1}\right]_{\geqslant 0}. \tag{21}$$

If we replace $x_n^{(k)}$ by $(1-q)x_n^{(k)}$ and take the limit $q\to 1$, the q-Sato equation (20) is reduced to the $\widehat{\mathfrak{gl}}_N$ hierarchy discussed in [1]. In this sense, we call as 'q–$\widehat{\mathfrak{gl}}_N$ hierarchy' the hierarchy described by (20).

Hereafter we restrict ourselves to the case $N = 3$, and set $x_n^{(k)} = 0$ for $n \geqslant 2$. We will use the abbreviation $x_k = x_1^{(k)}$, $T_k = T_q(x_k)$, $C_k = C_1^{(k)} (k = 1, 2, 3)$. Then we can rewrite the q-Sato equation (20) as

$$\{-zx_k I_k + V_k(\underline{x})\}\widetilde{W} = (T_k\widetilde{W})(-zx_k I_k + I), \tag{22}$$

where $\widetilde{W} = W$ or $\bar{W}$, and $V_k(\underline{x})$ is defined by

$$V_k(\underline{x}) = I - x_k\{(T_k W_1(\underline{x}))I_k - I_k W_1(\underline{x})\}. \tag{23}$$

The matrix $V_k(\underline{x})$ is related to $C_k(z;\underline{x})$ as

$$I - x_k C_k(z;\underline{x}) = -zx_k I_k + V_k(\underline{x}). \tag{24}$$

The concrete expressions of $V_k(z;\underline{x})(k = 1, 2, 3)$ are given as follows:

$$V_1(z;\underline{x}) = I - x_1\begin{bmatrix} T_1(w_{11}) - w_{11} & -w_{12} & -w_{13} \\ T_1(w_{21}) & 0 & 0 \\ T_1(w_{31}) & 0 & 0 \end{bmatrix}, \tag{25}$$

$$V_2(z;\underline{x}) = I - x_2\begin{bmatrix} 0 & T_2(w_{12}) & 0 \\ -w_{21} & T_2(w_{22}) - w_{22} & -w_{23} \\ 0 & T_2(w_{32}) & 0 \end{bmatrix}, \tag{26}$$

$$V_3(z;\underline{x}) = I - x_3\begin{bmatrix} 0 & 0 & T_3(w_{13}) \\ 0 & 0 & T_3(w_{23}) \\ -w_{31} & -w_{32} & T_3(w_{33}) - w_{33} \end{bmatrix}, \tag{27}$$

where $w_{ij} = w_{ij}(\underline{x})$ denotes the (i, j) element of W_1.

The q–$\widehat{\mathfrak{gl}}_3$ hierarchy contains a q-analogue of the three-wave resonant system. To see this, we consider the reduced case of (17), which does not depend on s:

$$\mathcal{D}_q(x_k)\Psi_q(z;\underline{x}) = C_k(z;\underline{x})\Psi_q(z;\underline{x}). \tag{28}$$

This can be rewritten as

$$T_k\Psi_q(z;\underline{x}) = \{-zx_k I_k + V_k(\underline{x})\}\Psi_q(z;\underline{x}). \tag{29}$$

From the compatibility condition of (29), we have

$$\begin{cases} x_k I_k V_l + x_l(T_l V_k)I_l = x_l I_l V_k + x_k(T_k V_l)I_k, \\ (T_k V_l)V_k = (T_l V_k)V_l, \end{cases} \tag{30}$$

for $k, l = 1, 2, 3$. Substituting (25), (26), (27) for (30), one obtains

$$\mathcal{D}_q(x_k)w_{ij} = (T_k w_{ik})w_{kj}. \tag{31}$$

If we impose the condition $w_{ji} = w_{ij}^*$ (complex conjugate of w_{ij}), equations (31) can be regarded as a q-analogue of the three-wave resonant system.

In what follows, the matrix $\bar{W}_0(\underline{x})$ plays a crucial role. We prepare several lemmas for latter use.

Lemma 1. *Under the reduction condition (18), $\bar{W}_0(\underline{x})$ satisfies $T_k \det \bar{W}_0(\underline{x}) = \det \bar{W}_0(\underline{x})$.*

Proof. The $\widehat{\mathfrak{gl}}_3$ q-Sato equation (20) implies

$$\{T_a W(z)\}(I - zx_a I_a)\{W(z)\}^{-1} = \{T_a \bar{W}(z)\}(I - zx_a I_a)\{\bar{W}(z)\}^{-1}. \tag{32}$$

It follows that

$$\det[T_a W(z)] \cdot \{\det W(z)\}^{-1} = \det[T_a \bar{W}(z)] \cdot \{\det \bar{W}(z)\}^{-1}. \tag{33}$$

There are no positive powers with respect to z on the left-hand side, while no negative powers on the right-hand side. Thus we obtain the result from the degree 0 term. □

Lemma 2. *The matrix $\bar{W}_0(\underline{x})$ satisfies $T_k \bar{W}_0(\underline{x}) = V_k(\underline{x})\bar{W}_0(\underline{x})$*

Proof. This is a direct consequence of (22) with $\widetilde{W} = \bar{W}$. □

Lemma 3. $\det[V_k + \lambda I_k] = 1 + \lambda (k = 1, 2, 3)$.

Proof. The case with $\lambda = 0$ follows from lemmas 1 and 2. Using this result, we have

$$\det[V_k + \lambda I_k] = \det[V_k] + \lambda \times \{\text{the } (k,k) \text{ co-factor of } V_k\} = 1 + \lambda,$$

where we have used (25), (26) and (27). □

3. Similarity reduction to q-Painlevé VI

3.1. Similarity reduction to q-Schlesinger system

Motivated by the scaling symmetry (proposition 1), we impose the following conditions on W and $\bar{W}$, which we call 'similarity conditions':

$$W(\mathrm{e}^{\partial_s}; s, \underline{x}) = q^{s+D(\alpha)} \circ W(\mathrm{e}^{\partial_s}; s, \underline{x}_q) \circ q^{-s-D(\alpha)}, \tag{34}$$

$$\bar{W}(\mathrm{e}^{\partial_s}; s, \underline{x}) = q^{s+D(\alpha)} \circ \bar{W}(\mathrm{e}^{\partial_s}; s, \underline{x}_q) \circ q^{-s-D(\beta)}. \tag{35}$$

Under the reduction condition (18), the similarity conditions (34), (35) take the form

$$W(z; \underline{x}) = q^{D(\alpha)} W(q^{-1}z; \underline{x}_q) q^{-D(\alpha)}, \tag{36}$$

$$\bar{W}(z; \underline{x}) = q^{D(\alpha)} \bar{W}(q^{-1}z; \underline{x}_q) q^{-D(\beta)}. \tag{37}$$

The similarity condition for $\bar{W}_0(\underline{x})$ follows from (37):

$$\bar{W}_0(\underline{x}) = q^{D(\alpha)} \bar{W}_0(\underline{x}_q) q^{-D(\beta)}. \tag{38}$$

We remark that the parameters $\alpha_i, \beta_i (i = 1, 2, 3)$ should obey the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3, \tag{39}$$

due to lemma 1.

The similarity conditions (36), (37) imply the following relation for $\Psi_q(z, \underline{x})$:

$$\Psi_q(qz, \underline{x}) = q^{D(\alpha)} \Psi_q(z, \underline{x}_q). \tag{40}$$

Applying (29), we can calculate $\Psi_q(z, \underline{x}_q)$ as

$$\Psi_q(z, \underline{x}_q) = \{-zx_1 I_1 + (T_2T_3V_1)\}\{-zx_2I_2 + (T_3V_2)\}\{-zx_3I_3 + V_3\}\Psi_q(z, \underline{x}). \tag{41}$$

Due to (22), one can rewrite this equation in two different ways:

$$\begin{aligned}\{-zx_1 I_1 &+ (T_2T_3V_1)\}\{-zx_2I_2 + (T_3V_2)\}\{-zx_3I_3 + V_3\}\\ &= (T_1T_2T_3W)\prod_{k=1}^{3}(I - zx_kI_k)W^{-1}\\ &= (T_1T_2T_3\bar{W})\prod_{k=1}^{3}(I - zx_kI_k)\bar{W}^{-1}.\end{aligned} \tag{42}$$

Since (42) has no negative powers with respect to z, we have

$$\begin{aligned}(T_1T_2T_3W)\prod_{k=1}^{3}(I - zx_kI_k)W^{-1} &= \left[(T_1T_2T_3W)\prod_{k=1}^{3}(I - zx_kI_k)W^{-1}\right]_{\geqslant 0}\\ &= \left[(T_1T_2T_3W)\left(I - z\sum_{k=1}^{3}x_kI_k\right)W^{-1}\right]_{\geqslant 0}\\ &= I - \sum_{k=1}^{3}x_k[z(T_1T_2T_3W)I_kW^{-1}]_{\geqslant 0}.\end{aligned} \tag{43}$$

It follows that (42) has the following expression:

$$(T_1T_2T_3\bar{W})\prod_{k=1}^{3}(I - zx_kI_k)\bar{W}^{-1} = U(\underline{x}) - z\sum_{k=1}^{3}x_kI_k, \tag{44}$$

where $U(\underline{x})$ is a 3×3 matrix. Comparing the z^0 terms in (44) and using (38), we get

$$U(\underline{x}) = (T_1T_2T_3\bar{W}_0)\bar{W}_0^{-1} = q^{-D(\alpha)}\bar{W}_0q^{D(\beta)}\bar{W}_0^{-1}. \tag{45}$$

Thus we have obtained a linear q-difference equation for Ψ_q:

$$\Psi_q(qz; \underline{x}) = \left\{-zq^{D(\alpha)}X + \bar{W}_0q^{D(\beta)}\bar{W}_0^{-1}\right\}\Psi_q(z; \underline{x}), \tag{46}$$

where $X = \mathrm{diag}[x_1, x_2, x_3]$. It is convenient to introduce a gauge-transformed function $\widetilde{\Psi}_q \overset{\mathrm{def}}{=} \bar{W}_0^{-1}\Psi_q$ that satisfies the following system of equations:

$$\widetilde{\Psi}_q(qz; \underline{x}) = \left\{-z\bar{W}_0^{-1}q^{D(\alpha)}X\bar{W}_0 + q^{D(\beta)}\right\}\widetilde{\Psi}_q(z; \underline{x}), \tag{47}$$

$$T_k\widetilde{\Psi}_q(z; \underline{x}) = \left\{-zx_k(T_k\bar{W}_0)^{-1}I_k\bar{W}_0 + I\right\}\widetilde{\Psi}_q(z; \underline{x}). \tag{48}$$

As we shall show in what follows, the system of the linear q-difference equations (47), (48) works as a Lax pair for the q-P_{VI} with (3×3)-matrix coefficients, which is a q-analogue of the formulation used in [13–15].

To establish a link between the (3×3)-matrix system (47), (48) and the (2×2)-matrix system (3), (4), we use a q-analogue of Laplace transform due to Hahn [16]. For a function $f(z)$, we define $\mathcal{L}_q[f](\zeta)$ and $\mathcal{L}_q^{-1}[f](z)$ as

$$\mathcal{L}_q[f](\zeta) = \frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{q^{-n}f(\zeta^{-1}q^{-n})}{(q^{-1}; q^{-1})_n}, \tag{49}$$

$$\mathcal{L}_q^{-1}[f](z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-n(n-1)/2} f(z^{-1}q^n)}{(q^{-1};q^{-1})_n}. \tag{50}$$

Transformations (49), (50) have the following properties:

$$\mathcal{L}_q[\mathcal{D}_{q^{-1}}(z)f(z)](\zeta) = \zeta\mathcal{L}_q[f(z)](\zeta) - (q^{-1};q^{-1})_\infty^{-1} f(0), \tag{51}$$

$$\mathcal{L}_q[zf(z)](\zeta) = \mathcal{D}_q(\zeta)\mathcal{L}_q[f(z)](\zeta), \tag{52}$$

$$\mathcal{L}_q^{-1}[\mathcal{D}_q(\zeta)f(\zeta)](z) = z\mathcal{L}_q^{-1}[f(\zeta)](z), \tag{53}$$

$$\mathcal{L}_q^{-1}[\zeta f(\zeta)](z) = \mathcal{D}_{q^{-1}}(z)\mathcal{L}_q^{-1}[f(\zeta)](z), \tag{54}$$

$$\mathcal{L}_q^{-1}[\mathcal{L}_q[f]](z) = f(z), \tag{55}$$

$$\mathcal{L}_q\big[\mathcal{L}_q^{-1}[f]\big](\zeta) = f(\zeta). \tag{56}$$

We outline a proof of (51)–(56) in the appendix.

If we define $\widetilde{\Phi}_q(z) = \mathcal{L}_q^{-1}[\widetilde{\Psi}_q(\zeta)](z)$, we can show that the transformed function $\widetilde{\Phi}_q(\zeta)$ satisfies the linear equations,

$$\mathcal{D}_{q^{-1}}(\zeta)\widetilde{\Phi}_q(\zeta;\underline{x}) = \sum_{j=1}^{3}\frac{\bar{W}_0^{-1}I_j\bar{W}_0(I - q^{D(\beta)+I})}{\zeta - q^{\alpha_j+1}x_j}\widetilde{\Phi}_q(\zeta;\underline{x}), \tag{57}$$

$$\mathcal{D}_q(x_k)\widetilde{\Phi}_q(\zeta;\underline{x}) = \frac{(T_k\bar{W}_0)^{-1}I_k\bar{W}_0(I - q^{D(\beta)+I})}{\zeta - q^{\alpha_k+1}x_k}\widetilde{\Phi}_q(\zeta;\underline{x}). \tag{58}$$

We can set $\beta_3 = -1$ without loss of generality. With this choice, we have $(I - q^{D(\beta)+I})_{j3} = 0 (j = 1, 2, 3)$ and we can restrict equations (57), (58) to the two-dimensional subspace $\{{}^t(\widetilde{\phi}_1, \widetilde{\phi}_2, 0)\}$. Thus we obtain the 2×2 system of the form

$$\mathcal{D}_{q^{-1}}(\zeta)\widetilde{Y}(\zeta;\underline{x}) = -\sum_{j=1}^{3}\frac{A_j(\underline{x})}{\zeta - q^{\alpha_j+1}x_j}\widetilde{Y}(\zeta;\underline{x}), \tag{59}$$

$$\mathcal{D}_q(x_k)\widetilde{Y}(\zeta;\underline{x}) = -\frac{B_k(\underline{x})}{\zeta - q^{\alpha_k+1}x_k}\widetilde{Y}(\zeta;\underline{x}), \tag{60}$$

where $A_k(\underline{x}), B_k(\underline{x}) (k = 1, 2, 3)$ are defined by

$$A_k(\underline{x}) = \begin{bmatrix}(\bar{W}_0^{-1})_{1k}\\(\bar{W}_0^{-1})_{2k}\end{bmatrix}[(\bar{W}_0)_{k1}(\bar{W}_0)_{k2}]\begin{bmatrix}q^{\beta_1+1}-1 & 0\\ 0 & q^{\beta_2+1}-1\end{bmatrix}, \tag{61}$$

$$B_k(\underline{x}) = \begin{bmatrix}((T_k\bar{W}_0)^{-1})_{1k}\\((T_k\bar{W}_0)^{-1})_{2k}\end{bmatrix}[(\bar{W}_0)_{k1}(\bar{W}_0)_{k2}]\begin{bmatrix}q^{\beta_1+1}-1 & 0\\ 0 & q^{\beta_2+1}-1\end{bmatrix}. \tag{62}$$

We remark that the matrices A_1, A_2, A_3 satisfy the relation

$$A_1 + A_2 + A_3 + I = \begin{bmatrix}q^{\beta_1+1} & 0\\ 0 & q^{\beta_2+1}\end{bmatrix}. \tag{63}$$

We call (59) and (60) the q-Schlesinger system since the limiting case $q \to 1$ coincides with the Schlesinger system associated with P_{VI}.

3.2. Relation with the q-Painlevé VI

Hereafter we set $\beta_3 = -1$, $x_3 = 0$. We introduce $Y(\zeta;\underline{x})$ as

$$Y(\zeta;\underline{x}) = \frac{(q^{\alpha_1}x_1\zeta^{-1};q^{-1})_\infty(q^{\alpha_2}x_2\zeta^{-1};q^{-1})_\infty}{(\zeta;q^{-1})_\infty^2(q^{-1}\zeta^{-1};q^{-1})_\infty^2}\widetilde{Y}(\zeta;\underline{x}). \tag{64}$$

From (59), (60), we have

$$Y(q^{-1}\zeta;\underline{x}) = \mathcal{A}(\zeta;\underline{x})Y(\zeta;\underline{x}), \tag{65}$$

$$T_kY(\zeta;\underline{x}) = \zeta^{-1}\{(\zeta - q^{\alpha_k+1}x_k)I + x_kB_k(x)\}Y(\zeta;\underline{x}), \tag{66}$$

where the coefficient matrix $\mathcal{A}(\zeta;\underline{x})$ is given by

$$\begin{aligned}\mathcal{A}(\zeta;\underline{x}) &= (\zeta - q^{\alpha_1+1}x_1)(\zeta - q^{\alpha_2+1}x_2)(I + A_3(x))\\ &\quad + \zeta(\zeta - q^{\alpha_2+1}x_2)A_1(x) + \zeta(\zeta - q^{\alpha_1+1}x_1)A_2(x).\end{aligned} \tag{67}$$

The coefficient matrix $\mathcal{A}(\zeta;\underline{x})$ has the form

$$\mathcal{A}(\zeta;\underline{x}) = \mathcal{A}_2\zeta^2 + \mathcal{A}_1\zeta + \mathcal{A}_0, \tag{68}$$

where the matrices $\mathcal{A}_k = \mathcal{A}_k(\underline{x})$ $(k = 0, 1, 2)$ are given by

$$\begin{aligned}&\mathcal{A}_0 = q^{\alpha_1+\alpha_2+2}x_1x_2(I + A_3), \qquad \mathcal{A}_2 = \mathrm{diag}[q^{\beta_1+1}, q^{\beta_2+1}],\\ &\mathcal{A}_1 = -(q^{\alpha_1+1}x_1 + q^{\alpha_2+1}x_2)\mathcal{A}_2 + q^{\alpha_1+1}x_1A_1 + q^{\alpha_2+1}x_2A_2.\end{aligned} \tag{69}$$

Proposition 2. *Eigenvalues of $\mathcal{A}_0$ are $x_1x_2q^{\alpha_1+\alpha_2+2}$, $x_1x_2q^{\alpha_1+\alpha_2+\alpha_3+3}$.*

Proof. Denote as $F(\lambda)$ the characteristic polynomial of $\mathcal{A}_0$:

$$F(\lambda) = \det[\lambda I - \mathcal{A}_0] = \det[\tilde{\lambda}I - q^{\alpha_1+\alpha_2+2}x_1x_2A_3], \tag{70}$$

where we have set $\tilde{\lambda} = \lambda - q^{\alpha_1+\alpha_2+2}x_1x_2$. Using the fact

$$\left(\bar{W}_0^{-1}I_3\bar{W}_0(q^{D(\beta)+I} - I)\right)_{ij} = \begin{cases}(A_3)_{ij} & (1 \leqslant i, j \leqslant 2),\\ 0 & (j = 3),\end{cases} \tag{71}$$

we can rewrite $\tilde{\lambda}F(\tilde{\lambda})$ in terms of a 3×3 determinant:

$$\begin{aligned}\tilde{\lambda}F(\tilde{\lambda}) &= \det\left[\tilde{\lambda}I - q^{\alpha_1+\alpha_2+2}x_1x_2\left\{\bar{W}_0^{-1}I_3\bar{W}_0(q^{D(\beta)+I} - I)\right\}\right]\\ &= \det\left[\tilde{\lambda}I - q^{\alpha_1+\alpha_2+2}x_1x_2I_3\left(\bar{W}_0q^{D(\beta)+I}\bar{W}_0^{-1} - I\right)\right].\end{aligned} \tag{72}$$

From (23) and (38) with lemma 2, it follows that

$$\begin{aligned}I_3\bar{W}_0q^{D(\beta)+I}\bar{W}_0^{-1} &= q^{D(\alpha)+I}I_3(T_1T_2\bar{W}_0)\bar{W}_0^{-1}\\ &= q^{D(\alpha)+I}I_3(T_2V_1)V_2\\ &= q^{D(\alpha)+I}\{I_3 - x_1I_3(T_1T_2W_1)I_1 - x_2I_3(T_2W_1)I_2\\ &\quad + x_1x_2I_3(T_1T_2W_1)I_1(T_2W_1)I_2\}.\end{aligned} \tag{73}$$

Thus we have $\left(I_3\bar{W}_0q^{D(\beta)+I}\bar{W}_0^{-1}\right)_{33} = q^{\alpha_3+1}$ and obtain

$$\tilde{\lambda}F(\tilde{\lambda}) = \tilde{\lambda}^2\{\tilde{\lambda} - q^{\alpha_1+\alpha_2+2}x_1x_2(q^{\alpha_3+1} - 1)\}, \tag{74}$$

which proves the proposition. □

Proposition 3. $\det[\mathcal{A}(\zeta;\underline{x})] = q^{\alpha_1+\alpha_2+\alpha_3+3}\prod_{j=1}^{2}(\zeta - x_j)(\zeta - q^{\alpha_j+1}x_j)$.

Proof. Due to (61) and (67), $\det[\mathcal{A}(\zeta;\underline{x})]$ can be written as a 3×3 determinant:

$$\begin{aligned}\det[\mathcal{A}(\zeta;\underline{x})] &= (\zeta - q^{\alpha_1+1}x_1)^2(\zeta - q^{\alpha_2+1}x_2)^2 \det\big[I + \bar{W}_0^{-1} I_3 \bar{W}_0 (q^{D(\beta)+I} - I)\\ &\quad + \zeta(\zeta - q^{\alpha_1+1}x_1)^{-1}\bar{W}_0^{-1} I_1 \bar{W}_0 (q^{D(\beta)+I} - I)\\ &\quad + \zeta(\zeta - q^{\alpha_2+1}x_2)^{-1}\bar{W}_0^{-1} I_2 \bar{W}_0 (q^{D(\beta)+I} - I)\big]\\ &= (\zeta - q^{\alpha_1+1}x_1)^2(\zeta - q^{\alpha_2+1}x_2)^2 \det\big[I + I_3 \bar{W}_0 (q^{D(\beta)+I} - I)\bar{W}_0^{-1}\\ &\quad + \zeta(\zeta - q^{\alpha_1+1}x_1)^{-1} I_1 \bar{W}_0 (q^{D(\beta)+I} - I)\bar{W}_0^{-1}\\ &\quad + \zeta(\zeta - q^{\alpha_2+1}x_2)^{-1} I_2 \bar{W}_0 (q^{D(\beta)+I} - I)\bar{W}_0^{-1}\big]. \end{aligned} \tag{75}$$

Applying the similarity condition (38) to (75), we have

$$\begin{aligned}\det[\mathcal{A}(\zeta;\underline{x})] &= q^{\alpha_1+\alpha_2+\alpha_3+3}\zeta^2(\zeta - q^{\alpha_1+1}x_1)(\zeta - q^{\alpha_2+1}x_2)\\ &\quad\times \det[(T_2 V_1)V_2 - \zeta^{-1}x_1 I_1 - \zeta^{-1}x_2 I_2].\end{aligned} \tag{76}$$

Furthermore, from (23), it follows that

$$(T_2V_1)V_2 - \zeta^{-1}x_1I_1 - \zeta^{-1}x_2I_2 = (T_2V_1 - \zeta^{-1}x_1I_1)(V_2 - \zeta^{-1}x_2I_2). \tag{77}$$

According to (76), (77) and lemma 3, we obtain the result. □

Next we consider the coefficient matrix of (66) with $k = 1$.

Lemma 4. $\det[(\zeta - q^{\alpha_1+1}x_1)I + x_1 B_1(x)] = (\zeta - x_1)(\zeta - q^{\alpha_1+1}x_1)$.

Proof. Due to (62), the determinant above can be written as a 3×3 determinant:

$$\begin{aligned}&\det[(\zeta - q^{\alpha_1+1}x_1)I + x_1 B_1(x)]\\ &\quad= (\zeta - q^{\alpha_1+1}x_1)^{-1}\det[(\zeta - q^{\alpha_1+1}x_1)I + x_1(T_1\bar{W}_0)^{-1} I_1 \bar{W}_0 (q^{D(\beta)+I} - I)]\\ &\quad= (\zeta - q^{\alpha_1+1}x_1)^{-1}\det\big[(\zeta - q^{\alpha_1+1}x_1)I + x_1 V_1^{-1} I_1\big(\bar{W}_0 q^{D(\beta)+I}\bar{W}_0^{-1} - I\big)\big]\\ &\quad= (\zeta - q^{\alpha_1+1}x_1)^{-1}\det\big[(\zeta - q^{\alpha_1+1}x_1)I + x_1 I_1\big\{q^{D(\alpha)+I}(T_1V_2) - V_1^{-1}\big\}\big],\end{aligned} \tag{78}$$

where we have used (38) in the final line. The result follows from a direct computation with (23). □

Now we are in position to state our main result.

Theorem 1. *Assume that $W(z;\underline{x})$ and $\bar{W}(z;\underline{x})$ solve the q-Sato equation (22), and satisfy the similarity conditions (36), (37) with $\beta_3 = -1$. Take $Y_q^{(\infty)}(\zeta;x_1,x_2)$ as the (2×2)-matrix-valued function associated with $W(z;\underline{x})$, and $Y_q^{(0)}(\zeta;x_1,x_2)$ with $\bar{W}(z;\underline{x})$. If we replace q by q^{-1} and set $x_1 = \gamma t$, then the functions*

$$Y^{(*)}(\zeta,t) = [Y_q^{(*)}(\zeta;\gamma t, x_2)]_{q\to q^{-1}} \quad (* = \infty, 0) \tag{79}$$

solve the q-difference system (3), (4). The parameters are identified as follows:

$$\begin{aligned}&\kappa_1 = q^{-\beta_1-1}, && \kappa_2 = q^{-\beta_2-1}, && \theta_1 = \gamma x_2 q^{-\alpha_1-\alpha_2-2}, && \theta_2 = \gamma x_2 q^{-\alpha_1-\alpha_2-\alpha_3-3},\\ &a_1 = \gamma, && a_2 = \gamma q^{-\alpha_1-1}, && a_3 = x_2, && a_4 = x_2 q^{-\alpha_2-1}.\end{aligned} \tag{80}$$

Proof . We have already proved that both $Y_q^{(\infty)}(\zeta;\underline{x})$ and $Y_q^{(0)}(\zeta;\underline{x})$ solve (65) with (69). The coefficient matrix $\mathcal{A}(\zeta;\underline{x})$ satisfies the desirous condition as shown in propositions 2 and 3. The remaining task is to rewrite (66) as (4). Using lemma 4 to calculate the inverse of the coefficient matrix of (66), we get

$$\tilde{Y}(\zeta;\underline{x}) = \frac{\zeta\{(\zeta - x_1 q^{\alpha_1+1})I + x_1\widetilde{B}_1(\underline{x})\}}{(\zeta - x_1)(\zeta - x_1 q^{\alpha_1+1})} T_1\tilde{Y}(\zeta;\underline{x}), \tag{81}$$

where $\widetilde{B}_1(\underline{x})$ is defined by

$$\widetilde{B}_1(\underline{x}) = \begin{bmatrix} (1-q^{\beta_2+1})(\bar{W}_0)_{12} \\ -(1-q^{\beta_1+1})(\bar{W}_0)_{11} \end{bmatrix} \left[\left(T_j \bar{W}_0^{-1}\right)_{21} - \left(T_j \bar{W}_0^{-1}\right)_{11} \right]. \tag{82}$$

Applying T_1^{-1} to (81), we obtain

$$T_1^{-1}\tilde{Y}(\zeta;x) = \frac{\zeta\left\{(\zeta - x_1 q^{\alpha_1})I + q^{-1}x_1\left(T_1^{-1}\widetilde{B}_1(\underline{x})\right)\right\}}{(\zeta - q^{-1}x_1)(\zeta - x_1 q^{\alpha_1})}\tilde{Y}(\zeta;x). \tag{83}$$

If we replace q by q^{-1} and set $\mathcal{B}_0 = -x_1 q^{-\alpha_1} I + q x_1 (T_1[\widetilde{B}_1]_{q\to q^{-1}})$, then equation (83) agrees with (4). □

Corollary 1. *Under the assumption of theorem 1, we can obtain a solution of the q-$P_{\rm VI}$ written in terms of $\bar{W}_0$:*

$$y = \left[-\frac{(\mathcal{A}_0)_{12}}{(\mathcal{A}_1)_{12}} \right]_{q\to q^{-1}}, \tag{84}$$

$$z = \left[\frac{\{(\mathcal{A}_0)_{12} + x_1(\mathcal{A}_1)_{12}\}\{(\mathcal{A}_0)_{12} + x_1 q^{\alpha_1+1}(\mathcal{A}_1)_{12}\}}{q(\mathcal{A}_1)_{12}\{(\mathcal{A}_0)_{11}(\mathcal{A}_1)_{12} - (\mathcal{A}_1)_{11}(\mathcal{A}_0)_{12}\} + q^{\beta_2+2}((\mathcal{A}_0)_{12})^2} \right]_{q\to q^{-1}}, \tag{85}$$

with

$$(\mathcal{A}_0)_{12} = q^{\alpha_1+\alpha_2+2}(q^{\beta_2+1}-1)x_1x_2\left(\bar{W}_0^{-1}\right)_{13}(\bar{W}_0)_{32}, \tag{86}$$

$$(\mathcal{A}_1)_{12} = (q^{\beta_2+1}-1)\left\{q^{\alpha_1+1}x_1\left(\bar{W}_0^{-1}\right)_{11}(\bar{W}_0)_{12} + q^{\alpha_2+1}x_2\left(\bar{W}_0^{-1}\right)_{12}(\bar{W}_0)_{22}\right\}, \tag{87}$$

$$(\mathcal{A}_0)_{11} = q^{\alpha_1+\alpha_2+2}x_1x_2\left\{1 + (q^{\beta_1+1}-1)\left(\bar{W}_0^{-1}\right)_{13}(\bar{W}_0)_{31}\right\}, \tag{88}$$

$$\begin{aligned}(\mathcal{A}_1)_{11} = &-q^{\beta_1+1}(q^{\alpha_1+1}x_1 + q^{\alpha_2+1}x_2) \\ &+ (q^{\beta_1+1}-1)\left\{q^{\alpha_1+1}x_1\left(\bar{W}_0^{-1}\right)_{11}(\bar{W}_0)_{11} + q^{\alpha_2+1}x_2\left(\bar{W}_0^{-1}\right)_{12}(\bar{W}_0)_{21}\right\}.\end{aligned} \tag{89}$$

4. Concluding remarks

In this paper, we have obtained the q-Painlevé VI (1) as a similarity reduction of the q–$\widehat{\mathfrak{gl}}_3$ hierarchy (20). Our method is a q-analogue of the (3×3)-matrix formulation of the Painlevé VI developed in [13–15]. The technique of the Laplace transform has been used to make a connection between a (2×2)-Fuchsian system and a 3×3 system with irregular singularities [14, 15]. To construct the q-analogue, we have used the q-Laplace transform (49), which was introduced in [16]. Note that similar but different versions of q-Laplace transformations have been discussed in several literature [17, 18].

We have constructed a class of solutions for the q-$P_{\rm VI}$ written in terms of $\bar{W}_0$ (corollary 1). Comparing to the results on the multi-component KP hierarchy (see, for example, [12]), it may be natural to introduce τ-functions in the following manner:

$$(\bar{W}_0(\underline{x}))_{ij} = \frac{\tau_{ij}(\underline{x})}{\tau(\underline{x})} \quad (i,j = 1,2,3). \tag{90}$$

However, this choice of the τ-functions seems to be different form that of [19, 20]. It may be important to clarify the relationship between the results in [19, 20] and the q–$\widehat{\mathfrak{gl}}_3$ hierarchy.

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Appendix

For reader's convenience, we outline a proof of the formulae for the q-Laplace transformation. Note that the parameter q is chosen as $0 < |q| < 1$ in [16], while $|q| > 1$ in this paper. In this appendix, we set $0 < |q| < 1$ in accordance with [16], and redefine $\mathcal{L}_q$ and $\mathcal{L}_q^{-1}$ as

$$\mathcal{L}_q[f](\zeta) = \frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{q^n f(\zeta^{-1}q^n)}{(q;q)_n}, \tag{A.1}$$

$$\mathcal{L}_q^{-1}[f](z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2} f(z^{-1}q^{-n})}{(q;q)_n}. \tag{A.2}$$

If we replace q by q^{-1}, (A.1) and (A.2) coincide with (49) and (50), respectively.

Proposition 4. *Transformation (A.1) has the property*

$$\mathcal{L}_q[\mathcal{D}_q(z)f(z)](\zeta) = \zeta\mathcal{L}_q[f(z)](\zeta) - (q;q)_\infty^{-1} f(0). \tag{A.3}$$

Proof. We introduce a truncated version of $\mathcal{L}_q$ as

$$\mathcal{L}_q^{(M)}[f](\zeta) = \frac{1}{\zeta}\sum_{n=0}^{M}\frac{q^n f(\zeta^{-1}q^n)}{(q;q)_n}. \tag{A.4}$$

Then we have

$$\begin{aligned}\mathcal{L}_q^{(M)}[\mathcal{D}_q(z)f(z)](\zeta) &= \sum_{n=0}^{M}\frac{f(\zeta^{-1}q^n) - f(\zeta^{-1}q^{n+1})}{(q;q)_n}\\ &= \sum_{n=0}^{M}\frac{f(\zeta^{-1}q^n)}{(q;q)_n} - \sum_{n=0}^{M}(1-q^{n+1})\frac{f(\zeta^{-1}q^{n+1})}{(q;q)_{n+1}}\\ &= \sum_{n=0}^{M}\frac{q^n f(\zeta^{-1}q^n)}{(q;q)_n} - (1-q^{M+1})\frac{f(\zeta^{-1}q^{M+1})}{(q;q)_{M+1}}.\end{aligned} \tag{A.5}$$

Taking the limit $M \to \infty$, we obtain formula (A.3). □

Formula (A.3) coincides with (51) by replacing q by q^{-1}. The remaining formulae (52)–(54) can be obtained in similar manner.

Proposition 5. *Transformations (A.1), (A.2) satisfy* $\mathcal{L}_q^{-1}[\mathcal{L}_q[f]](z) = f(z)$.

Proof. A straightforward calculation shows that

$$\mathcal{L}_q^{-1}[\mathcal{L}_q[f]](z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j q^{i+j(j+1)/2}}{(q;q)_i (q;q)_j} f(xq^{i+j})$$
$$= \sum_{k=0}^{\infty} q^k f(xq^k) \sum_{j=0}^{k} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_{k-j}(q;q)_j}. \tag{A.6}$$

The result follows from the formula

$$(z;q)_k = \sum_{j=0}^{k} \frac{(q;q)_k}{(q;q)_j (q;q)_{k-j}} (-z)^j q^{j(j-1)/2}, \tag{A.7}$$

by setting $z = 1$. □

Relation (56) can be proved in the same fashion.

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